CALCULUS EDUCATION

Toward a Pump, Not a Filter

by Billy Goodman

Calculus, the mathematics of change, is one of the great endeavors of the human mind. Calculus, the course, is one of the great tormentors of human minds. Taught around the country to more than one-half million college students every year, calculus in the classroom resembles the following idealization.

The professor knows that. He or she writes some rules for it on the blackboard, the students momentarily perk up and dutifully copy them.

The students know this; to solve a problem, they look for the example closest to the homework problem. The homework may or may not be collected and graded. Their answers can be checked at the back of the book.

In a typical class the homework assignment is to find a dozen derivatives or to calculate integrals. Most vary only slightly from the examples in the text. The students know this; to solve a problem, they look for the example closest to the homework problem. The homework may or may not be collected and graded. Their answers can be checked at the back of the book.

On examinations the students again see variations of the template problems, with a few harder ones to challenge the good students. This time the students cannot look back in the book; the course rewards those with good memories for the techniques.

Despite the fact that calculus teachers generally teach what experience has shown students can do, students fail the course at rates that approach 50 percent at some universities. Beyond the examination grade, the professor may provide no feedback.

Full of Recipes

It is no wonder that mathematicians refer contemptuously to calculus as a cookbook course, full of recipes. As described by many experienced teachers, cookbook calculus is a bland diet. Rules and algorithms replace concepts. Rote replaces understanding. The course is driven by examinations (two or three test scores may be all that determine a student's grade).

Calculus is a required subject for most majors in the sciences, engineering, and, of course, mathematics. It is also a prerequisite for admission to many graduate schools. Some departments and professional schools find that calculus is a useful filter, reducing the number of applications they have to consider and often serving as a reasonable predictor of student success. Even students who escape calculus unscathed, however, often do not display the mathematical reasoning skills that science and engineering professors want.

Keith Schwingendorf, who teaches calculus at Purdue University, tells a story that applies, save for the details, to almost any college or university. A mechanical engineering professor mentioned in passing to a class of sophomores that an integral is a sum. He simply assumed that the students had learned this basic idea from first-year calculus. But the students stared uncomprehendingly back at the professor. "Students seem to have a facility for doing things," Schwingendorf concludes, "but they lack a sense of ideas."

It is just such a sense of ideas that Paul Zorn of St. Olaf College in Northfield, Minnesota, and his colleagues are trying to instill in their students. The St. Olaf mathematicians have been working several years to improve calculus teaching. At least one fact suggests that the course was already better than average: More than 100 students graduate with mathematics majors each year at St. Olaf—from a class of only 750. Zorn attributes that in part to a department that "doesn't regard the study of mathematics as an activity for the elite" and assigns its best classroom teachers—including Zorn—to calculus I sections of about 30 students.

Nevertheless, Zorn and his colleagues have noticed the stagnation of calculus over the years. "For at least ten years," he says, "there has been the feeling that the course has been reduced to drills rather than focused on ideas like rates of change. What does the derivative tell you? What does the integral tell you? These ideas have not been emphasized."

The basic ideas of calculus, he notes, are just as important as they ever were, if not more so, in today's increasingly technical world.

"The manipulations of calculus," Zorn says, "are ready-made for testing. That lets a person teach a course that seems to be about calculus but in effect is just about symbol manipulation. It's much harder to teach a course that gets at the ideas behind the symbols."

Indeed, teaching the traditional course is, as Ohio State University's Bill Davis delicately puts it, "very comfortable for the faculty." He estimates the time a teacher puts into calculus at class hours plus 10 percent. Other mathematicians were even less generous, including one who said lectures could be prepared in the elevator on the way to class.

About five years ago a ground swell began among mathematicians to improve calculus teaching. Now many ex-
Efforts by mathematicians and educators to reform the way calculus is taught to undergraduates take into account changes both in the demographics and the technologies of mathematics education.

Zooming in. To understand the geometric meaning of a derivative at Smith, where Jim Callahan (left) teaches, students zoom in on a computer graph, magnifying smaller and smaller portions of a region until it approximates a straight line.
periments are underway that incorporate changes in pedagogy and content and that bring calculus finally into the computer era. Nonetheless, the question remains: How did the calculus course evolve to its present, widely despised, state?

How come?

Mathematicians who took calculus before the 1960s describe a leisurely paced course, usually taken during the sophomore year. The audience was fairly homogeneous—professors could count on their students having had four years of high school mathematics, high school physics, and then freshman courses in algebra and analytic geometry. "When we did the second derivative," says Gail Young, a visiting professor of mathematics education at Teachers College of Columbia University, who took calculus in 1935, "it cast light on the physics—acceleration. The teacher didn't have to explain what that was about." Class size was small and the only students taking calculus, for the most part, were intending to major in physical sciences or engineering (few majored in mathematics; calculus then, as now, was primarily a service course).

Then came Sputnik. Americans, it was thought, were behind the Soviets technologically. An effort was mounted to catch up. Funds flowed into scientific research and education and into mathematics. College enrollments mushroomed. Rigor, in the form of precise language and theory, invaded the calculus curriculum.

The change in emphasis from a more intuitive presentation might have worked well, if only the same highly motivated, well-prepared students had been taking calculus. But many more students were enrolling in calculus; young professors in the life sciences and social sciences were doing quantitative work and wanted their students to take calculus. For example, in 1963 four out of 71 business schools required differential calculus for admission. By 1966 the number had jumped to 38 of 84 schools.

Large lectures for calculus, once rare, became more common. In a background paper prepared for a pathbreaking 1987 conference, "Calculus for a New Century," Richard D. Anderson of Louisiana State University reported that 38 percent of calculus students were enrolled in classes of fewer than 40 students; 41 percent were in lectures of more than 80. Very few mathematicians have kind words for large lectures. Don Lewis, a mathematician at the University of Michigan, wonders whether one can teach students to converse and speak in a new language—calculus—in such large classes. "You wouldn't think of doing it in French or German," he says.

As calculus enrollments swelled, it could no longer be taken for granted that all students would have the same background of high school mathematics and science. Calculus became a freshman course, contributing to the demise of college-level precalculus courses. Calculus teaching changed to accommodate the increasing variety of increasingly poorly prepared students. More time was spent reviewing precalculus material and teaching techniques for solving standardized problems. Calculus teachers "felt pressure to pass students, and so devised a curriculum and tests that students could pass," says Duke University's David Smith, who has been teaching calculus since the early 1960s. Once considered a desirable teaching assignment, calculus became less so as more money became available to mathematicians for research and as calculus became more of a production-line course.

Weighty texts

The massive textbook typical of today—the size and weight of a cinder block—had its origin during the 1960s. The big sellers are all longer than 1,000 pages. Fifty years ago the average calculus book was 250 to 300 pages, wrote Murray Protter, author of a well-known calculus text and a mathematician at the University of California at Berkeley, in The Mathematical Intelligencer. The size of the books grew in part because they included more of the background that used to be taken for granted. To support teachers who were emphasizing techniques, the books included more and more worked-out examples and end-of-chapter exercises. (Some popular books boast of more than 6,000 exercises.)

The uniformity of textbooks (almost all have nearly identical tables of contents) makes the teaching of anything but a traditional course difficult. Only a few mavericks will go to the trouble of designing and teaching a course without textbook support. Textbooks that deviate from the norm tend not to last.

In 1976 Peter Lax of the Courant Institute of Mathematical Sciences at New York University wrote a radical book, Calculus with Applications and Computing, with two colleagues, Samuel Burstein and Anneli Lax. In the preface they said that the 500-page book's purpose is to emphasize the relation of calculus to
science. Peter Lax says the book was intended to reflect the notion that scientists and applied mathematicians, when faced with a problem that calculus can help them solve, go directly to a computer. Lax also believes that computing will increase students' understanding of calculus concepts.

For example, students spend a lot of time in a traditional course learning a half-dozen or so techniques of integration, most of which work only with a highly restricted set of functions. (Most integrals cannot be obtained in so-called closed form, that is, as a simple algebraic expression.) But "students misunderstand integrals," Lax says, "because they mix them up with all the no-good techniques for getting them." Computers show the integral for what it is: a number representing the area under a curve. Numerical techniques, such as the Riemann sum method of adding up thin rectangles to approximate the area under the curve, work for any integral and are perfectly suited to the iterative powers of a computer.

Lax's textbook was not widely adopted. Publishers and teachers felt it was too difficult or ahead of its time.

At many colleges and universities, textbooks are adopted by committee. Committees tend to be a force for conservatism; they try to reach a consensus on a book. They look for familiar topics.

Some mathematicians believe that the more topics a book contains, the more likely it is to be adopted. This practice selects for larger books. Also, schools may be hesitant to adopt an unusual book until it has a track record, but it cannot get a track record if no one is adopting it.

**Rumblings**

During the 1970s, at a few isolated outposts, mathematicians taught calculus in idiosyncratic ways, often distributing their own notes to students rather than relying heavily on a textbook. An example of one such course, which lasted to become a model for a current reform effort, was offered at Hampshire College in Amherst, Massachusetts. Hampshire was created in 1970 by its four neighbors—the University of Massachusetts and Amherst, Smith, and Mt. Holyoke colleges—to be a place where experiments in education could be tried. (Collectively, the institutions are called the Five Colleges.) Grades, majors, and departments are nonexistent at Hampshire. Each student develops his or her own program of study in consultation with faculty members.

In the late 1970s Hampshire students complained to mathematicians Kenneth Hoffman and Michael Sutherland that they were not getting the mathematical tools they needed in order to read and understand papers in their disciplines, especially ecology and economics. Hoffman and Sutherland looked through journals in these fields and were surprised, Hoffman says.

They found calculus was used far less often than statistics. Where they did find calculus, they were finding not such dreaded Calculus I (or II) topics as integration by parts or l'Hôpital's rule, but systems of differential equations, traditionally a topic left to calculus III or IV. As Hoffman puts it, "Calculus was a language and \( \frac{dy}{dx} \) had to feel like a natural way of talking."

Hoffman began teaching calculus with emphasis on differential equations and the scientific context for calculus. He also made heavy use of computers, which permit a numerical, as opposed to algebraic, approach to solving differential equations or integrals. The Five Colleges experimental course—Calculus in Context—owes much to the spirit of the course developed at Hampshire.

The complacency of the calculus-teaching community was jolted from outside on a much wider scale in the early 1980s by a group of mathematicians and computer scientists. Led by Anthony Ralston, a computer scientist at the State University of New York at Buffalo, they argued that, in this era of computers, calculus no longer deserved to be the only core mathematics course for all college students. Some, especially those who would study computer science, would be better off studying discrete mathematics, including such topics as sets and relations, formal logic, graph theory, counting and probability. The reasons given ranged from the mathematical (some kinds of ideas, such as recursive thinking and induction, are better handled in discrete mathematics than in calculus) to the applied (computer science and some other disciplines make greater use of discrete mathematics than they do of calculus).

**Discrete mathematics**

Ralston's challenge was not an attack on calculus. "I didn't even know much about calculus teaching at the time," he says. It was a plea for discrete mathematics to play a role more or less equal to calculus in the mathematics training of freshmen and sophomores. Nevertheless, the questioning of calculus' status as the core mathematics course for college students stimulated mathematicians to think about the importance of calculus. Some colleges and universities
began experimenting with discrete mathematics instead of, or in addition to, calculus.

Ronald Douglas, vice-provost for undergraduate studies at the State University of New York at Stony Brook, who was then chairman of the mathematics department, faced what would amount to a fundamental change in departmental teaching. After thinking about the issues and talking with colleagues, Douglas decided that calculus was as important as ever. As he wrote in Toward a Lean and Lively Calculus, the first manifesto of the calculus reform movement, "Almost all of science is concerned with the study of systems that change, and the study of change is the very heart of the differential calculus. . . . Even discrete analysis of such equations using finite differences is next to meaningless without . . . calculus. Thus, all science and engineering students need calculus in their studies."

Douglas organized two conferences that set the agenda for calculus reform. The first, sponsored by the Sloan Foundation and held at Tulane University in January 1986, involved just 25 invited participants. They discussed the forces that have shaped the teaching of calculus, from textbooks to the exploding target population, and the failure of calculus instruction to utilize calculators and computers. They concluded that it was time for a change. Three workshops on content, methods, and implementation made specific recommendations. Toward a Lean and Lively Calculus, the conference report, was widely read and discussed.

The second conference, "Calculus for a New Century," held in October 1987, reached a much broader audience. Sponsored by the National Academies of Sci-

Lean and lively textbooks

Calculus textbooks are big business. A new one costs more than $1 million to produce and requires several years of writing, testing, and revising. The top books sell more than 40,000 copies a year, providing their authors with a comfortable return and publishing companies with comfortable profits.

Almost no one has kind words for the 1,100-page, 6,000-problem, five-pound textbooks that lead the market today. But the committees that choose textbooks at colleges and universities are buying them. When experimental texts have been tried in the past, says Steve Quigley, senior mathematics editor at the Boston-based publisher PWS-KENT, the mathematics community has not bought them. "I'm frustrated as an editor," says Quigley. "I want to publish a book that the market deems credible, but there are so many hurdles."

Publishers are afraid to publish radical, revolutionary books, for example, because they are not likely to be adopted widely enough to defray the expense. Publishers are prepared, Quigley says, to put out experimental books emerging from calculus reform projects if the books are produced inexpensively. That will allow the publishers to break even on smaller sales. Thus, new books are likely to have one color instead of the fashionable two or four. They may even be put out between soft covers instead of hard.

The first such lean and lively texts are already hitting the market. Donald Small and John Hosack of Colby College came out with what may have been the first, in the spring of 1990, Calculus: An Integrated Approach (McGraw-Hill). Tufts University and Whittier College, among other places, have adopted it. Undeniably leaner (685 pages, including appendices), the text places heavy emphasis on concepts and treatment of the calculus of several variables right from the start, Small says.

Gilbert Strang of the Massachusetts Institute of Technology and Thomas Dick of Oregon State University each published a new book in early 1991. Strang, a well-regarded mathematician and author of a popular linear algebra textbook, says he was strongly influenced by the "Calculus for a New Century" meeting in Washington in 1987 to try a fresh approach to calculus even though he was not heavily involved in calculus teaching himself. A textbook "is something one person can do—it's not like changing the curriculum across the country," he says.

Strang calls his Calculus a mainstream book that gently encourages computer use (especially graphics) and offers a more conceptual treatment than the traditional text. Since he publishes his own books, he recognizes as well as anyone that a radical treatment would not be easily accepted. "Most revolutions end at two pi," is how he describes that constraint. "Calculus is well established," he adds. "If you sweep everything away, it's sure not to be followed."

Strang was aiming for a lively style and fewer than 700 pages of text (it has, in fact, 615). His book includes iterations, which are easy to do with computers.

Several calculus reformers have produced drafts of books that are used in their classrooms. Typically, the manuscripts are photocopied and distributed to the students. One of the most ambitious at this stage is Calculus by the Harvard Core Calculus Consortium. The consortium includes faculty members from Harvard University, Suffolk County Community College, the University of Arizona, Stanford University, Chelmsford (Massachusetts) High School, Haverford College, the University of Southern Mississippi, and Colgate University. The goal of the diverse group, says Harvard's Deborah Hughes Hallett, the consortium coordinator, is to produce a textbook that could be adopted nationwide, in a variety of school settings.

The book treats functions with the "Rule of Three"—analytically, numerically, and graphically. Scientific functions are treated early. Applications are drawn from an extremely broad array of sources, in contrast to the physics-heavy applications of a standard course. Exercises are frequently designed to be worked with a calculator and often require explanations as answers. • B. G.
ence and Engineering, the conference attracted more than 600 mathematicians, scientists, educators, and publishers to Washington, D.C., to continue the rethinking of the calculus course. The subtitle of the published proceedings, *A Pump, Not a Filter*, alludes to a goal that emerged from the conference: to make calculus so attractive and successful that it would no longer filter students out of the science pipeline but rather encourage study of science and provide the necessary skills in quantitative reasoning. At the conference and in reports and readings prepared for the proceedings, scientists suggested what they wanted from calculus—more real-life examples, more modeling, more connection with discrete ideas, more approximate and numerical methods, and more use of computing to achieve these goals.

**The ground swell**

The discussions at the conference and the published proceedings provided ammunition for calculus teachers throughout the country to return to their departments and press for change. They could point to the proceedings as evidence that they were not alone. Informal talk of calculus and calculus teaching—long limited among teachers to plaintive laments about the quality of students—advanced to constructive discussion of techniques and theories of calculus teaching and learning. Edward Gaughan, who is working at New Mexico State University on a revitalized course that features difficult, extended projects, says that, since the conference, “there’s lots of talk about calculus reform. You can’t go to a mathematics conference without a packed session on it.”

There is more than talk, however. Experimental courses are sprouting up at colleges, universities, and even some high schools. For the most part, experimental sections are running side by side with traditional sections.

**New ways**

One of the most noteworthy features of virtually every revitalized calculus course is that students are given more responsibility for their own learning. In many instances they are encouraged and guided to discover the mathematics for themselves. The process starts by sharply deemphasizing the traditional lecturing technique.

At the University of Illinois, for example, Jerry Uhl and Horacio Porta have...
simply eliminated the lecture. The entire course is delivered on computer, using an electronic textbook composed of Mathematica, the software, and interactive student notebooks. Students work through problems presented in Mathematica, using the program’s symbolic and graphing abilities, and add their answers, queries, and explanations directly to the text.

During the fall of 1988, when Uhl and Porta first offered their new course, they continued to give lectures because, as Uhl admits, “we thought that was what was expected of us.” The lectures ended abruptly when students stopped showing up for them. They were in line for the computers.

Although he has all but eliminated lecture in his experimental course, Ed Dubinsky of Purdue University says his students still come to a classroom. The classes feature much more give and take, however. Dubinsky or his colleague Keith Schwingendorf frequently pose a mathematical issue to the class, and say, “take a minute to talk about this.” The students sit in groups of four, and turn to each other to talk mathematics—an unusual occurrence in a traditional class.

One day, Dubinsky says, he stopped class "to try to tie down a notion and explain a bit more. But as I began my explanation, one student raised his hand and started giving the explanation for me. Another student continued where the first left off. The class wouldn’t let me lecture.” That shows, he says, that once students have had a taste of being responsible for their own learning, they are reluctant to allow a return to the old ways, however briefly.

At the University of Illinois computer laboratory (“Home of the calculating Illini” says the sign), students who might have skipped lectures are now sometimes asked to leave when the lab-

---

**Calculus problems, then and now**

In the cookbook calculus texts that dominated the 1980s, exercises at the end of the chapter took the form: “Find dy/dx in exercises 1 through 18,” or “Evaluate the integrals in exercises 19 through 34.”

A favorite kind of word problem, one that could be counted on to be part of any examination, is called the “related rate” problem. An example (from Calculus and Analytic Geometry by George Thomas, or just “Thomas” to the hundreds of thousands of students who have used this classic text):

“A man 6 feet tall walks at the rate of 5 ft/sec toward a street light that is 16 feet above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 feet from the base of the light?”

Harvard’s Deborah Hughes Hallett says those are “artificial problems that don’t have much use elsewhere. In a related rates problem, you find a rate of change by differentiating. But in the real world, you usually know the rates and want to know the final state, so you use differential equations. Students are learning algorithms they’ll never apply.”

What sorts of problems are students doing in reformed calculus courses? A sampling follows, showing the emphasis on modeling, data, geometrical understanding, and use of technology.

**Student Research Project, Calculus—NMSU**

Your parents are going to knock out the bottom of the entire length of the south wall of their house and turn it into a greenhouse by replacing some bottom portion of the wall with a huge sloped piece of glass (which is expensive). They have already decided they are going to spend a certain fixed amount. The triangular ends of the greenhouse will be made of various materials they already have lying around.

The floor space in the greenhouse is only considered usable if they can both stand up in it, so part of it will be unusable, but they don’t know how much. Of course, that depends on how they configure the greenhouse. They want to choose the dimensions of the greenhouse to get the most usable floor space in it, but they are at a real loss to know what the dimensions should be and how much usable space they will get. Fortunately, they know you are taking calculus. Amaze them.

**Calculus in Context—Five Colleges**

The following examples are from the first chapter of the Calculus in Context textbook (students learn about simultaneous differential equations—called “rate equations” in the text—before they have seen a formal definition of derivative):

1. Suppose the spread of an illness similar to measles is modeled by the following rate equations:

   \[
   S' = -0.0002 SI
   I' = 0.0002 SI - 0.8 I
   R' = 0.8 I
   \]

   (a) Roughly how long does someone who catches this illness remain infected? Explain your reasoning.

   (b) Suppose 100 people in the population are currently ill. How large does the population have to be before the illness “takes off”—that is, before there is an increase in the number of cases? Explain your reasoning.

   (c) According to the model, how many (of the 100 infected) will recover during the next 24 hours?
Laboratory closes at 11 p.m. One student still logged on at 6 p.m. said he liked the course because "I can play around with the functions and I can do things quickly that would take half an hour by hand."

Mathematica is indeed a powerful symbol manipulation program, but its purpose is not just to permit students to do standard problems quickly. It is there to foster a conjectural approach to calculus, to allow students to play with functions by changing parameters and seeing what happens. "Every example is infinitely many examples," says Uhl.

The goal, according to Porta, is to have students "announce" the theory after working through carefully plotted examples and examining graphs. For example, says Uhl, "a standard course would define an odd function, then verify that a bunch of examples are odd. We start with the functions." Students plot a variety of both odd and even functions, learn the differences, and come up with their own definitions.

The computer is also being used to encourage students to construct mathematics for themselves. At Purdue, for instance, Dubinsky is a leading spokesman for a small but growing group of mathematicians who believe that computer programming activities can enhance understanding of calculus.

Programming

Dubinsky is also one of the few mathematicians actually carrying out a research program on the ways students learn calculus. His message is that students do not learn spontaneously—by listening to a lecture—or inductively—by working examples. Mathematics knowledge is something you do, not something you have, he asserts.

In an address to St. Olaf College's conference on calculus in October 1989, Dubinsky said: "Using a software package that performs some mathematical operations is another way of showing the mathematics to the user. Writing computer code that implements a mathematical process or represents a mathematical object is a construction by the programmer." When students make the appropriate constructions on a computer, Dubinsky believes, they will construct the appropriate images in their minds, such as the important idea that functions can be thought of as processes or objects.

Students have difficulty seeing functions as objects, he says, and thus often fail to appreciate that the derivative of a function is itself a function. Dubinsky and his colleagues have students write programs in which functions are both inputs and outputs. Instead of numbers, the objects that students expect, the programs return functions.

Traditionally, the mathematics community has been shy about using programming as a teaching tool, according...
to Arnold Ostebee of St. Olaf College. There are several reasons for that. Some mathematicians argue that there is barely enough time to teach calculus, let alone teach programming, too. And some believe that students will become entangled in the syntax of programming language and that they will lose sight of the mathematics.

Another potential problem was described by Lawrence Moore of Duke University. “We’ve found in programming that students do it minimally,” he says. “That is, they write a program and use error messages to fix it until it runs.” In other words, students don’t necessarily have to understand what they are doing, if the programming language they are using is helpful enough. They simply fiddle with the program one piece at a time until it runs.

Dubinsky believes these problems can be solved, and he is studying the possibility that a language he helped develop, ISETL (which stands for Interactive Set Language), may work. To those who say that learning to program takes time, Dubinsky has argued that that is not necessarily true: ISETL minimizes the “frustrating syntax issues that are not connected with the mathematical issues,” making learning easier and more natural, he feels. Further, Duke University’s Smith agrees, the programming activities that are part of the Purdue course require thought and are not likely to be completed merely by fiddling with the program.

Exportability

One unsolved problem of experimental computer-based courses is how to disseminate them more widely. The University of Illinois computer laboratory has 40 Macintoshes, each of which can be used by about five students a semester. Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory. Can the courses be delivered to more than a few hundred students per school? Purdue has a similar laboratory.

With computers and calculators able to perform the algorithms that formerly constituted most of the calculus course, it becomes senseless to ask students to do the same sorts of tasks traditionally asked of them in calculus. Now the premium is on helping students learn to set up problems, recognize that, for example, an integration is involved, and tell the computer what to calculate. For instance, as Columbia University’s Gail Young explains, “to get a maximum of a function, the fastest way is often to pull out your HP-28C and zoom in on the graph. You wouldn’t use the traditional method of taking the first derivative and setting it equal to zero.”

Mathematics for science

Not everything new about calculus courses is related to technology. At Duke University students often collect data themselves, making the course much closer in spirit to a science laboratory than a traditional calculus class. For one project, students used a door, knob and a length of string to measure the way the period of a pendulum varies with the length of the string. They could compare their results with the numerical solution of the differential equation that models pendulum motion that they worked out in their computer laboratory. “We discovered that students like doing things with their hands,” says Lang Moore.

By such experimentation, students learn that in the real world of science, as opposed to the idealized world of the traditional calculus course, functions are rarely presented as neat, algebraic expressions. Instead, scientists, economists, and engineers have to deal with messy data or intuitively drawn curves. Since they are the ones who have to use calculus the most, the course should reflect their needs.

Jim Callahan of Smith College, for instance, explains it this way: “Our clientele is here to get training for science. We’ve abandoned the orthodox mathematician’s view of the course and adopted the view of scientists. Moreover, since calculus grew out of scientific problems, it is our view that this is the best training for future mathematicians as well.”

Not only does this view mean messy data and implicit functions, it means in-

Zorn. Going beyond symbols.
tuition rather than rigor. Proofs in the mathematical language of epsilons and deltas do not belong in such a course.

Judging from a traditional course, students might assume that the most important functions in science are polynomials. But these are just the functions that students can handle most easily. One of the goals of calculus reformers, according to Deborah Hughes Hallett of Harvard University, one of the leaders of the Harvard Core Calculus Consortium, is to shift the palette of functions being used in the course. "If you look at the functions most frequently applied—in biology, physics, engineering—they are linear, logarithmic, and exponential," Hughes Hallett says. "But our students come having had lots of polynomials. The functions you want them to know the best are the ones they find the scariest, because in the mad dash to get to calculus in high school, they rush through them."

Extended projects

At New Mexico State University at Las Cruces, calculus reform is proceeding without resort to technology and without necessarily adopting a particularly applied flavor. The principal difference in the NMSU course is the use of extended projects, which count for a significant part of a student's grade. The projects are smaller versions of problems that might be given to graduate students, says Douglas Kurtz, one of the five principal investigators on the project. They require that students create something new.

At first, to many of the students, the projects seem impossible. One difficult project, called "Algebra of Infinite Limits," requires students to arrive at their own definitions of limit without resorting to intuitive arguments learned in class. Then they must use the definition they have developed to make and prove additional conjectures about limits.

The students have two weeks to complete a project and are encouraged to seek help from the faculty and do independent reading. Since the projects are so difficult, they must start early and discuss the problem intensively with others. Students learn, says sophomore Wes Berger, that "if you see a problem that's huge, you take it apart into smaller problems."

The genesis of the projects approach came when two faculty members, David Pengelley and Marcus Cohen, were, as Pengelley recalls, "bitching and moaning, as mathematicians do, about our students: 'They're not motivated; they don't spend enough time on math.'"

"We've all had the experience," Pengelley continues, "of having students come to our offices and say, 'Dr. Pengelley, I couldn't study for your math exam because I had a big project in another course and spent three weeks on it.' We began to wonder why we couldn't give projects that would take our students time."

During the last few years many mathematicians at New Mexico State have helped write projects, which now number more than 100 and will be published shortly as an instructor's book. The book will state each project, the mathematics it requires, and what students can learn from it.

It is too soon to evaluate the success of calculus reform. Most calculus is still taught in the traditional way. And most experimental courses have been in existence only a year or two. Their success must be measured by means other than the traditional exam on algorithms (although where students in experimental sections have taken a common final with students in traditional sections, no differences have emerged).

How good?

Mathematicians, and the client departments, would like the students, as a result of the exposure to calculus, to become good problem solvers and be able to deal with functions defined by data, translate word problems into mathematical language and then use computers and calculators to solve them, make some headway on a difficult problem even without technological help, and explain a calculus concept or other quantitative idea in plain language.

Mathematicians involved in calculus reform are clearly excited again about teaching. And their students, by and large, recognize that something new is going on and appreciate it, even if they do not always like the extra work demanded of them.

New Mexico State's Pengelley, after four years of student research projects, is convinced the program is working. "Our students weren't giving us their best before on rote problems," Pengelley says. "They're achieving more because we're challenging them intellectually. We now have much higher opinions of our students."

The National Science Foundation contributes to the support of work discussed in this article primarily through its Division of Undergraduate Science, Engineering, and Mathematics Education and its Division of Mathematical Sciences.